Generalized Pencil Of Function Method

In recent times, the methodology of approximating a function by a sum of complex exponentials has found applications in many areas of electromagnetics. For example in antenna pattern synthesis [1] and in the extraction of the s-parameters of microwave integrated circuits [2], in the computation of input impedance of electrically wide slot antennas [3] and in the efficient evaluation of the Sommerfeld integral [4], that is our goal. Two of the most popular linear methods used to estimate the parameters of a sum of complex exponentials are the polynomial method, i.e Prony method, and the matrix pencil method (called also GPOF)[5]. This method differs from the well known Prony method because it finds the poles by solving a generalized eigenvalue problem instead of the conventional two-step process where the first step involves the solution of a matrix equation, and the second step entails finding the roots of a polynomial, as is required by Prony method.

Formulation

Let us consider a generic signal y(t)

$$y(t) = x(t) + n(t) \approx \sum_{i=1}^{M} R_i e^{s_i t} + n(t)$$
 (1)

where n(t) is the possible noise of the system. We sample y(t) at kT_s , where $k = 0, 1, \dots, N-1$ and T_s is the sampling period obtaing

$$y(kT_s) = x(kT_s) + n(kT_s) \approx \sum_{i=1}^{M} R_i z_i^k + n(kT_s)$$
 (2)

where

$$z_i = e^{s_i T_s} = e^{(\alpha_i + j\omega_i)T_s} \tag{3}$$

where $i = 1, 2, \dots, M$. Let us consider first a noiseless signal y(t)

$$y(t) = x(t) \approx \sum_{i=1}^{M} R_i e^{s_i t}$$
(4)

sampling this signal we obtain

$$y(kT_s) = x(kT_s) \approx \sum_{i=1}^{M} R_i z_i^k$$
(5)

with

$$z_i = e^{s_i T_s} = e^{(\alpha_i + j\omega_i)T_s} \tag{6}$$

where i = 1, 2, ..., M. Our goal is to estimate M, s_i, R_i that approximate the function y(t)

We define two matrices $(N - L) \times L \underline{Y}_1, \underline{Y}_2$

$$\underline{Y}_{=1} = \begin{bmatrix}
y(0) & y(1) & \cdots & y(L-1) \\
y(1) & y(2) & \cdots & y(L) \\
\vdots & \vdots & \vdots & \vdots \\
y(N-L-1) & y(N-L) & \cdots & y(N-2)
\end{bmatrix}_{(N-L)\times L}$$
(7)
$$\underline{Y}_{=2} = \begin{bmatrix}
y(1) & y(2) & \cdots & y(L) \\
y(2) & y(3) & \cdots & y(L+1) \\
\vdots & \vdots & \vdots & \vdots \\
y(N-L) & y(N-L+1) & \cdots & y(N-1)
\end{bmatrix}_{(N-L)\times L}$$
(8)

Introducing

$$\underline{\underline{R}} = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & R_m \end{bmatrix}$$
(9)
$$\begin{bmatrix} z_1 & 0 & \cdots & 0 \end{bmatrix}$$

$$\underline{\underline{Z}_{0}} = \begin{bmatrix} \vdots & \vdots & \vdots \\ 0 & z_{2} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & z_{m} \end{bmatrix}$$
(10)

$$\underline{\underline{Z}}_{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_{1} & z_{2} & \cdots & z_{M} \\ \vdots & \vdots & & \vdots \\ z_{1}^{N-L-1} & z_{2}^{N-L-1} & \cdots & z_{M}^{N-L-1} \end{bmatrix}_{(N-L)\times M}$$
(11)

$$\underline{\underline{Z}}_{2} = \begin{bmatrix} 1 & z_{1} & \cdots & z_{1}^{L-1} \\ 1 & z_{2} & \cdots & z_{2}^{L-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_{M} & \cdots & z_{M}^{L-1} \end{bmatrix}_{M \times L}$$
(12)

 $\underline{\underline{Y}}_{1}, \underline{\underline{Y}}_{2}$ can be written as

$$\underline{\underline{Y}}_{1} = \underline{\underline{Z}}_{1} \underline{\underline{R}} \underline{\underline{Z}}_{2} \tag{13}$$

$$\underline{\underline{Y}}_{2} = \underline{\underline{Z}}_{1} \underline{\underline{R}} \underline{\underline{Z}}_{0} \underline{\underline{Z}}_{2}$$
(14)

Now let us consider the matrix pencil

$$\underline{\underline{Y}}_{2} - \lambda \underline{\underline{Y}}_{1} = \underline{\underline{Z}}_{1} \underline{\underline{R}} \left\{ \underline{\underline{Z}}_{0} - \lambda \underline{\underline{I}} \right\} \underline{\underline{Z}}_{2}$$
(15)

where \underline{I} is a $M \times M$ identity matrix. We can demostrate that if $M \leq L \leq N - M$ the rank of $\underline{Y}_2 - \lambda \underline{Y}_1$ is M; however, if $\lambda = z_i \ i = 1, 2, \dots M$ the i-th row of $\underline{Z}_0 - \lambda \underline{I}_1$ is zero and the rank of $\underline{Y}_2 - \lambda \underline{Y}_1$ is reduced to M - 1. Moreover, the poles are the generalized eigenvalues of the matrix pencil, so the problem of solving for z_i can be cast as an ordinary eigenvalue problem. Let us define the Moore–Penrose pseudo inverse matrix of \underline{Y}_1 as

$$\underline{\underline{Y}}_{1}^{+} = \{\underline{\underline{Y}}_{1}^{H}\underline{\underline{Y}}_{1}\}^{-1}\underline{\underline{Y}}_{1}^{H}$$

$$\tag{16}$$

where the superscript h indicates the conjugate transpose. Then,

$$\underline{\underline{Y}}_{1}^{+}\left[\underline{\underline{Y}}_{2}-\lambda\underline{\underline{Y}}_{1}\right] = \underline{\underline{Y}}_{1}^{+}\underline{\underline{Z}}_{1}\underline{\underline{R}}\left[\underline{\underline{Z}}_{0}-\lambda\underline{\underline{I}}\right]\underline{\underline{Z}}_{2}$$
(17)

$$= \left[\underline{\underline{Y}}_{1}^{H}\underline{\underline{Y}}_{1}\right]^{-1}\underline{\underline{Y}}_{1}^{H}\underline{\underline{Y}}_{2} - \left[\underline{\underline{Y}}_{1}^{H}\underline{\underline{Y}}_{1}\right]^{-1}\underline{\underline{Y}}_{1}^{H}\lambda\underline{\underline{Y}}_{1}$$
(18)

$$= \left[\underline{\underline{Y}}_{1}^{H}\underline{\underline{Y}}_{1}\right]^{-1}\underline{\underline{Y}}_{1}^{H}\underline{\underline{Y}}_{2} - \lambda \underline{\underline{I}}$$
(19)

$$= \underline{\underline{Y}}_{1}^{+} \underline{\underline{Y}}_{2} - \lambda \underline{\underline{I}}$$

$$\tag{20}$$

Thus the parameters z_i can be determined by solving the generalized eigenvalue problem of

$$\underline{\underline{Y}}_{1}^{+}\underline{\underline{Y}}_{2}^{-} \lambda \underline{\underline{I}}$$
⁽²¹⁾

In presence of noise, $(M \leq L)$ we define the matrix $\underline{\underline{Y}}$ as a combination of $\underline{\underline{Y}}_1, \underline{\underline{Y}}_2$

$$\underline{\underline{Y}} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{L-1} \\ y_1 & y_2 & \cdots & y_L \\ \vdots & \vdots & & \vdots \\ y_{N-L-1} & y_{N-L} & \cdots & y_{N-2} \end{bmatrix}_{(N-L)\times(L+1)}$$
(22)

$$\underline{\underline{Y}}_{(N-L)\times(L+1)} = \underline{\underline{U}} \underline{\underline{\Sigma}} \underline{\underline{V}}^{H}$$
(23)

where $\underline{U}, \underline{V}$ are unitary matrices with dimensions $(N-L) \times (N-L) \in (L+1) \times$ (L+1), respectively given by the eigenvalues of $\underline{\underline{Y}}\underline{\underline{Y}}^H$ and $\underline{\underline{Y}}^H\underline{\underline{Y}}$. While $\underline{\underline{\Sigma}}$ is a $(N-L) \times (L+1)$ diagonal matrix composed of the singular values of $\underline{\underline{Y}}$:

$$\underline{\underline{\Sigma}} = diag(\sigma_1, \sigma_2, \dots, \sigma_p); \qquad p = min\{N - L, L +\}$$
(24)

where $|\sigma_1| \ge |\sigma_2| \ge \ldots \ge |\sigma_p| \ge 0$.

Now we can estimate the M parameter. This is done comparing the largest singular value σ_{max} with the other σ_c and defining a threshold q. This is linked to the significant decimal digits in the data. Therefore, we consider only the singular values that are

$$\frac{|\sigma_c|}{|\sigma_{max}|} \approx 10^{-q} \tag{25}$$

The other singular values are treated as zero, since we consider them only given by noise. We can now construct a filtered matrix $\underline{V'}$ so that it contains only the M dominant right singular vectors of $\underline{\underline{V}}$ so that $\underline{\underline{Y}}$ is given by

$$\underline{\underline{Y}} = \underline{\underline{U}} \underline{\underline{\Sigma}'} \underline{\underline{V}'}^H \tag{26}$$

 $\underline{\underline{\Sigma}'}$ is containg only the M dominant singular values of $\underline{\underline{\Sigma}}$. We have defined $\underline{\underline{Y}}_1$ and $\underline{\underline{Y}}_2$ that are constructed deleting respectively the last and the firts row of $\underline{\underline{Y}}_1$

$$\underline{\underline{Y}}_{1} = \underline{\underline{U}} \underline{\underline{\Sigma}'} \underline{\underline{V}'}_{1}^{H}$$

$$\tag{27}$$

$$\underline{\underline{Y}}_{2} = \underline{\underline{U}} \underline{\underline{\Sigma}'} \underline{\underline{V}'}_{2}^{H}$$

$$\tag{28}$$

we can demostrate that the eigenvalues of

$$\left\{\underline{\underline{Y}}_{1}^{+} \, \underline{\underline{Y}}_{2}^{-} - \lambda \underline{\underline{I}}\right\} \tag{29}$$

are the same as the ones of

$$\{\underline{\underline{V_1}}^{H}\}^+ \{\underline{\underline{V_2}}^{H}\}^+ - \lambda \underline{\underline{I}}$$
(30)

Once the M and z_i are known we can solve the matrix equation by least square to calculate R_i .

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_m \\ z_1^2 & z_2^2 & & z_m^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \dots & z_m^{N-1} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_M \end{bmatrix}$$
(31)

Prony-Type Method

For sake of completeness, we describe also the Prony method, even if its computationally less efficient and also unstable in case of noisy signal. We consider once again the same generic signal y(t).

Given M complex number z_i , with i = 1, 2..., M, there exist only one complex number $a_i \in C$ with i = 1, 2, ...M, so that z_i are solutions of

$$\sum_{k=0}^{M} a_k z^{-k} = 0 \qquad a_0 \equiv 1$$
(32)

 \rightarrow finding the coefficient a_i is equivalent to determine the poles z_i . Moreover, we can demostrate that if

$$p(\lambda) = \sum_{k=0}^{L} a_k \lambda^{-k} = 0$$
(33)

where $p(z_i) \equiv 0 \forall i = 1, 2, ...M$ If

$$y_n = \sum_{i=1}^M R_i z_i^n \tag{34}$$

for n = 0, 1, .., N - 1 then, if $L \le m \le N - 1$ with $M \le L$

$$\sum_{k=0}^{L} y_{m-k} a_k = 0 \tag{35}$$

Considering

$$y_{m-k} = \sum_{i=1}^{M} R_i z^{m-k}$$
(36)

substituting

$$\sum_{k=0}^{L} y_{m-k} a_k = \sum_{k=0}^{L} \left(\sum_{i=1}^{M} R_i z_i^m z_i^{-k} \right) a_k = \sum_{i=1}^{M} R_i z_i^m \sum_{k=0}^{L} z_i^{-k} a_k = 0$$
(37)

$$p(z_i) \equiv 0 \tag{38}$$

We can write (35) in matrix form $\forall m \in (l, N-1)$

$$\underline{\underline{Y}a} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{L-1} & | & y_L \\ y_1 & y_2 & \cdots & y_L & | & y_{L+1} \\ \vdots & \vdots & \vdots & | & \vdots \\ y_{N-L-1} & \cdots & \cdots & y_{N-2} & | & y_{N-1} \end{bmatrix} \begin{bmatrix} a_L \\ a_{L-1} \\ \vdots \\ a_0 \end{bmatrix} = 0$$
(39)
$$\underline{\underline{Y}a} = \underline{\underline{Y}}_1 \underline{\underline{y}} \begin{bmatrix} \underline{a} \\ \cdots \\ a_0 \end{bmatrix}$$
(40)

 $a_0 \equiv 1 \Longrightarrow \underline{\underline{Y}a} = \underline{\underline{Y}}_1 \underline{\underline{a}} + \underline{\underline{y}} = 0$ so, since

 $\implies \underline{\underline{Y}}_{1}\underline{\underline{a}} = -\underline{\underline{y}}.$ where $\underline{\underline{Y}}_{1}$ has dimension of $L \times [(N-1) - L]$. In order to generalize the method we have to write the polynomial form of the signal so that we can distinguish the roots of the signal to those due to noise. This solution, called minimum-norm solution is given by

$$\underline{a} = -\underline{\underline{Y}}_{1}^{+}\underline{\underline{y}} \tag{41}$$

where $\underline{\underline{Y}}_{=1}^+$ is once again the Moore-Penrose pseudo-inverse matrix. In case of noisy signal the Moore-Penrose pseudo-inverse matrix $\underline{\underline{Y}}_{1}^{+}$ is replaced by a truncated rank-M pseudo-inverse which is formed by the first M largest singular values. The choice of M is the same as in the Matrix pencil method.

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